Due Fri

1.2 – Gaussian Elimination

7. Solve the system by Gaussian elimination.

$$x - y + 2z - w = -1$$
$$2x + y - 2z - 2w = -2$$
$$-x + 2y - 4z + w = 1$$
$$3x - 3w = -3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

1 2 3

$R_2 \to R_2 - 2R_1$

$$\begin{array}{ccccc} 2 & 1 & -2 & -2 & -2 \\ -2 & 2 & -4 & 2 & 2 \\ \hline 0 & 3 & -6 & 0 & 0 \end{array}$$

$R_3 \to R_3 + R_1$

$$\begin{array}{ccccc} -1 & 2 & -4 & 1 & 1 \\ 1 & -1 & 2 & -1 & -1 \\ \hline 0 & 1 & -2 & 0 & 0 \end{array}$$

$R_4 \to R_4 - 3R_1$

$$\begin{array}{ccccc} 3 & 0 & 0 & -3 & -3 \\ -3 & 3 & -6 & 3 & 3 \\ \hline 0 & 3 & -6 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

Then

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in row echelon form via Gaussian elimination.

The system is now $\begin{aligned} x - y + 2z - w &= -1 \\ y - 2z \quad &= 0 \end{aligned}$

We can solve by substitution:

$y = 2z$, $x - 2z + 2z - w = -1$

$x - w = -1 \Rightarrow x = w - 1$

Let $s = z$, $t = w$. Then

$$\boxed{x = t - 1, \ y = 2s, \ z = s, \ w = t}$$

Row echelon form (#1-3 below) and **reduced row echelon form** (#1-4 below) of a matrix – to be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is a **leading 1.**
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

Back to the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\begin{array}{ccccc} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & \boxed{0} & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x - w = -1 \Rightarrow x = w - 1 \\ \to y - 2z = 0 \Rightarrow y = 2z \\ \\ \\ \end{array}$$

This is in reduced row echelon form (rref)

via Gauss-Jordan elimination.

These are called leading 1s

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

They correspond to <u>leading variables</u>

pivot columns

Entries in the original matrix corresponding to leading 1s are pivots

Leading variables correspond to the leading 1's in the augmented matrix for a linear system.

Free variables are the remaining variables in the linear system.

In a **homogeneous linear system**, the constant terms are all zero. That is, it has the form

$$\begin{array}{cccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & 0 \end{array}$$

Theorem 1.2.1 Free Variable Theorem for Homogeneous Systems
If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem 1.2.2 A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Pf: The # of equations is the # of rows in the augmeted matrix for the system. The number of leading 1s is no more than the # of rows. The matrix for the system has more variable columns than rows, so the system has at least one free variable. Hence the system has infinitely many solutions. ✓

28. What condition, if any, must a, b, and c satisfy for the linear system to be consistent?

$$x + 3y + z = a$$
$$-x - 2y + z = b$$
$$3x + 7y - z = c$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\begin{array}{cccc} -1 & -2 & 1 & b \\ 1 & 3 & 1 & a \\ \hline 0 & 1 & 2 & a+b \end{array}$$

$R_3 \to R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 7 & -1 & c \\ -3 & -9 & -3 & -3a \\ \hline 0 & -2 & -4 & c-3a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & c-3a \end{array}\right]$$

1

$R_3 \to R_3 + 2R_2$

$$\begin{array}{cccc} 0 & -2 & -4 & c-3a \\ 0 & 2 & 4 & 2a+2b \\ \hline 0 & 0 & 0 & c+2b-a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & \boxed{0} & 0 & c+2b-a \end{array}\right] \rightarrow 0 = c + 2b - a$$

2

The system is consistent if $a = 2b + c$.

Solve the given linear system ~~by any method~~. using an (augmented) matrix

16.

Homogeneous ↓

$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

writing these zeros is optional

$$\begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$ * Then $R_2 \to \frac{1}{3}R_2$

$$\begin{array}{ccc} -2 & 4 & -6 \\ 2 & -1 & -3 \\ \hline 0 & 3 & -9 \end{array}$$

* Not an elementary row operation

$R_3 \to R_3 + R_2$

$$\begin{array}{ccc} 1 & 1 & 4 \\ -1 & 2 & -3 \\ \hline 0 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$ Then $R_3 \to \frac{1}{10}R_3$

$$\begin{array}{ccc} 0 & 3 & 1 \\ 0 & -3 & 9 \\ \hline 0 & 0 & 10 \end{array}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$\begin{array}{ccc} 2 & -1 & -3 \\ 0 & 0 & 3 \\ \hline 2 & -1 & 0 \end{array}$$

$R_2 \to R_2 - R_3$

$$\begin{array}{ccc} 0 & 1 & -3 \\ 0 & 0 & 3 \\ \hline 0 & 1 & 0 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_2$ then $R_1 \to \frac{1}{2}R_1$

$$\begin{array}{ccc} 2 & -1 & 0 \\ 0 & 1 & 0 \\ \hline 2 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$(x, y, z) = (0, 0, 0)$$

This is the trivial solution.

19.

$$
\begin{aligned}
2x + 2y + 4z &= 0\\
w - y - 3z &= 0\\
2w + 3x + y + z &= 0\\
-2w + x + 3y - 2z &= 0
\end{aligned}
$$

$R_1 \leftrightarrow R_2$, then $R_2 \to \frac{1}{2}R_2$

$$
\begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}
$$

1 2 3

$R_3 \to R_3 + R_4$

$$
\begin{array}{rrrr}
2 & 3 & 1 & 1 \\
-2 & 1 & 3 & -2 \\ \hline
0 & 4 & 4 & -1
\end{array}
$$

$R_4 \to R_4 + 2R_1$

$$
\begin{array}{rrrr}
-2 & 1 & 3 & -2 \\
2 & 0 & -2 & -6 \\ \hline
0 & 1 & 1 & -8
\end{array}
$$

$$
\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 4 & -1 \\ 0 & 1 & 1 & -8 \end{bmatrix}
$$

$R_3 \to R_3 - 4R_2$

$$
\begin{array}{rrrr}
0 & 4 & 4 & -1 \\
0 & -4 & -4 & -8 \\ \hline
0 & 0 & 0 & -9
\end{array}
$$

$R_4 \to R_4 - R_2$

$$
\begin{array}{rrrr}
0 & 1 & 1 & -8 \\
0 & -1 & -1 & -2 \\ \hline
0 & 0 & 0 & -10
\end{array}
$$

then $R_3 \to -\frac{1}{9}R_3$ $\quad 0\ 0\ 0\ 1$

$R_4 \to -\frac{1}{10}R_4$ $\quad 0\ 0\ 0\ 1$

$R_4 \to R_4 - R_3$ $\quad 0\ 0\ 0\ 0$

$$
\begin{bmatrix} 1 & 0 & -1 & \boxed{-3} \\ 0 & 1 & 1 & \boxed{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to
\begin{array}{cccc|c}
w & x & y & z & \\
\end{array}
\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

R_1: $w - y = 0$

R_2: $x + y = 0$

R_3: $z = 0$

w, x, z are leading variables.

y is a free variable.

Let $t = y$. Then

$$
\boxed{\begin{aligned} w &= t \\ x &= -t \\ y &= t \\ z &= 0 \end{aligned}}
$$

23. The augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

a. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$ consistent, unique solution

b. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$ consistent, infinitely many solutions

c. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow 0=1 \rightarrow$ inconsistent

d. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right]$ If this is $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{array}\right]$ consistent

If this is $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \leftarrow z=0 \\ \leftarrow z=1 \end{array}$

This is a contradiction

inconclusive